

Repelling Cycles Revisited

Thm. If f has degree d , then f has at most $6d-6$ attracting or indifferent cycles

↳ Test class: # attracting + # parabolic $\leq 2d-2$

Thm. (Shishikura) At most $2d-2$ attracting or indifferent.

Lemma. The number of indifferent cycles of multiplier $\lambda \neq 1$ is at most $4d-4$.

Proof.

Let $z_{1,0}, \dots, z_{k,0}$ be k distinct indiff. cycles.

1) $f(z) = \frac{p(z)}{q(z)}$, $p, q \in \mathbb{C}[z]$

$$f_t(z) = \frac{p(z) - t \cdot z^d}{q(z) - t}, \quad f_0(z) = f(z), \quad f_\infty(z) = z^d$$

is of deg d for $t \in \hat{\mathbb{C}} \setminus \mathbb{R}$, when $|z| < \infty$. (Resultant)

if $f_0^k(z_{i,0}) = z_{i,0}$, $\lambda_i = (f^k)'(z_{i,0}) \neq 1$, $|\lambda_i| \neq 1$. Then

$\frac{\partial}{\partial z} \Big|_{\substack{z=z_{i,0} \\ t=0}} (f_t^k(z) - z) \neq 0$, so by implicit function

theorem, $\exists$ $z_{i,t}$ has defined/open set of t near 0,

s.t. $f_t^k(z_{i,t}) = z_{i,t}$ and $\frac{\partial}{\partial z}(f_t^k(z) - z) \neq 0$.

2) Claim. The multipliers $t \mapsto \lambda_{i,t} = \frac{\partial}{\partial z} \big|_{z=z_{i,t}} (f_t^k(z))$ are nonconstant near $t=0$.

• If $\lambda_{i,t} = \lambda_i \neq 1$, $f_t^k(z_{i,t}) = z_{i,t}$, and $t \rightarrow t'$, then

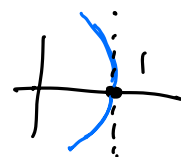
$\lambda_{i,t'} = \lim_{t \rightarrow t_0} \lambda_{i,t} = \lambda_i \neq 1$, $f_{t'}^k(z_{i,t'}) = z_{i,t'}$, so the set of

t for which $z_{i,t}$ exists is also closed in $\hat{\mathbb{C}} \setminus \mathbb{R}$.

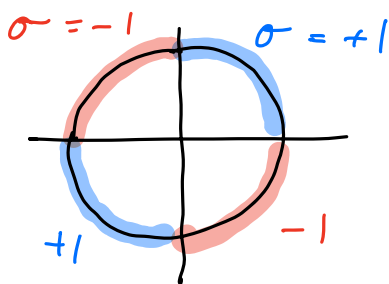
Open + closed $\Rightarrow$ all of $\hat{\mathbb{C}} \setminus \mathbb{R} \ni \infty$.

• But then this implies that $f_\infty(z) = z^d$ has a cycle of multipliers $|\lambda_i| = 1$.

3) Expand $\lambda_{i,t} = \lambda_i + b_i t^{u_i} + \dots$



Since nonconst. $|\lambda_{i,t}| = 1 + \operatorname{Re}(a_i t^{u_i}) + \dots$



$$\sigma_j(\theta) = \operatorname{sgn} \operatorname{Re}(a_j e^{2\pi i \theta u_j})$$

$\sigma_j > 0, < 0$ are u_j sectors

• If K is odd, let $\Sigma(\theta) := \sum_{j=1}^K \sigma_j(\theta)$, and

if K is even, $\Sigma(\theta) = \sum_{j=1}^{K-1} \sigma_j(\theta)$. Then

Σ is odd a.e. and $\int_0^1 \Sigma(\theta) d\theta = \sum_j \int_0^1 \sigma_j(\theta) d\theta$.

• So $\exists \theta$ s.t. $\sigma_j(\theta) = -1$ for more than half

of the term in the sum for $\Sigma(\theta)$. For r small, each $z_j, re^{i\theta}$ will be attractive.

• K odd: $\frac{K+1}{2} \leq 2d-2 \Rightarrow K \leq 4d-4$.

• K even: $\frac{K}{2} \leq 2d-2 \Rightarrow$

Prop. f has as many repelling cycles

Proof. Will show $\exists$ p -cycle for any large prime p . Suppose that f has no p -cycles.

Then any sol to $\underbrace{f^p(z) = z}_{d^p + 1 \text{ solutions not mult}} \Rightarrow \underbrace{f(z) = z}_{d+1 \text{ sol. not mult}}.$

Rigorous $\Rightarrow f^p(z) - z$ has a greater degree than $f(z) - z$ at some z_0 .

$f(z) = \lambda z + a z^{u+1} + \dots$, u is the degree of unity
 $\Rightarrow f^p(z) = \lambda^p z + \lambda^{p-1} a (1 + \lambda^u + \dots + \lambda^{u(p-1)}) z^{u+1} + \dots$

1) $\frac{\partial}{\partial z} (f^p(z) - z) = \lambda^p - 1 = 0$
 $\quad \quad \quad \underbrace{\text{with } f(z_0) = z_0}_{z_0}$

2) Need the leading exponent of $f^p(z)$ after z to be $> u+1$, so $\lambda \neq 1$.

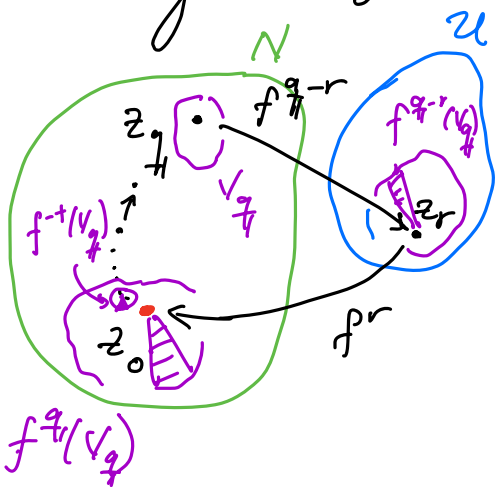
$\Rightarrow \lambda$ is a p^{th} root of unity.

• One of the fixed pts is periodic or mult p^{th} root of unity. Repeat for larger p until we see a root

f fixed pts.

Thm. The Julia set is the closure of the set of repelling periodic points.

Proof. 1) z_0 be a repelling fixed point, U be an open $U \cap J \neq \emptyset$.



2) $\exists z_r \in U$ st. $f^r(z_r) = z_0$

3) Let $N \ni z_0$ be a neighborhood where the Koenigs theorem holds.

$\exists z_q \in N$ st. $f^{q-r}(z_q) = z_r, z_q \neq z_0$.

By Koenigs, $z_{q+i} \in N$ st. $f^i(z_{q+i}) = z_q, z_{q+i} \rightarrow z_0$ Koenigs orbit

4) $\exists V_q \ni z_q$ st. $\overline{f^{q-r}(V_q)} \subset U, \overline{f^q(V_q)} \subset N$.

5) For t large, $z_{q+t} \in f^{-t}(V_q) \subset f^q(V_q)$.

If $(f^{q+t})'(z_{q+t}) \neq 0$, then $\exists! \bigcup_{z_{q+t}} \xrightarrow{f^{q+t}} f^{q+t}(w) \subset f^q(V_q)$

If $(f^{q+t})'(z_{q+t}) = 0$, then pick a wedge $W \subset f^{-t}(V_q)$ st. $f^{q+t}: W \xrightarrow{\sim} f^q(V_q)$.

6) $f^q(V_q) \xrightarrow{f^{-(q+t)}} W \hookrightarrow f^q(V_q)$ is strictly

contracting, $\exists$ fixed pt. p st. $f^{q+t}(p) = p$.

7) $f^{t-r}(p) \in U$ is periodic

Strengthening prev. results

* Recall: $\forall U, U \cap J \neq \emptyset, V = \bigcup_{n=1}^{\infty} f^n(U) \supset J$

Cor. if U is an open set and $U \cap J \neq \emptyset$, then for some $n, f^n(U) \supset J$.

Proof. $\exists$ repelling $z_0 \in U \cap J(f)$, wlog $f(z_0) = z_0$.

$\exists$ Koenigs neighborhood $N \subset U, N \ni z_0$, s.t. $N \subset f(N)$.

Then $\bigcup_{n=1}^{\infty} f^n(N)$ covers J . J is compact, so

$\Rightarrow J \subset f^n(N) \subset f^n(U)$.

* Attracting and parabolic attracted critical orbits.

Prop. if z_0 is a Cremer point or $z_0 \in \partial D$ for D a Siegel disk, then every neighborhood of z_0 contains as many critical orbit points.

Proof. 1) Cremer: $P = \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f))$, $\bar{P}$ is full-invariant, $\hat{\mathbb{C}} \setminus \bar{P}$ is backwards-invariant, $\hat{\mathbb{C}} \setminus \bar{P}$ is postcritical set.

$f: \hat{\mathbb{C}} \setminus f^{-1}(\bar{P}) \rightarrow \hat{\mathbb{C}} \setminus \bar{P}$ is a covering map.

$z_0 \in \hat{\mathbb{C}} \setminus \bar{P}, f(z_0) = z_0, \exists U \subset \hat{\mathbb{C}} \setminus \bar{P}, U \ni z_0$.

Choose branches: $f^{-n}: \underbrace{U}_{\text{loop}} \rightarrow \underbrace{\hat{\mathbb{C}} \setminus \bar{P}}_{\text{loop}}$

Assume $\Rightarrow f^{-u} \rightarrow g$ loc. unif. If g is const, then f^{-u} are constant, so z_0 is not indiff, so $g(u)$ is open.

• Pick $U' \subset g(u)$. Then

$$f^k(U') \subset f^k(g(u)) = \lim_{n \rightarrow \infty} f^{k+n}(u) \subset U \text{ for } k.$$

By Cor., $g(u) \ni z_0$ is in $F(f)$, but $z_0 \in J$.

2) Siegel disk: $z_0 \in \partial D$, $z_0 \in \overline{P}$.

• For $U \subset \hat{\mathbb{C}} \setminus \overline{P}$, $U \ni z_0$, $f^{-u}: U \rightarrow \hat{\mathbb{C}} \setminus \overline{P}$,



Assume. On D , f is an inv. rot, so we can choose $f^{-u} \rightarrow \text{id}$ on $U \cap D$ and U .

• Pick $U' \subset U$, $z_0 \in U'$, then $f^k(U') \subset U$ for any k .
concludes that $z_0 \in J$.